Jacquet modules of principal series generated by the trivial K-type

Noriyuki Abe

Abstract. We propose a new approach for the study of the Jacquet module of a Harish-Chandra module of a real semisimple Lie group. Using this method, we investigate the structure of the Jacquet module of principal series representation generated by the trivial K-type.

§1. Introduction

Let G be a real semisimple Lie group. By Casselman's subrepresentation theorem, any irreducible admissible representation U is realized as a subrepresentation of a certain non-unitary principal series representation. Such an embedding is a powerful tool to study an irreducible admissible representation but the subrepresentation theorem dose not tell us how it can be realized.

Casselman [Cas80] introduced the Jacquet module $J(U)$ of U. This important object retains all information of embeddings given by the subrepresentation theorem. For example, Casselman's subrepresentation theorem is equivalent to $J(U) \neq 0$. However the structure of $J(U)$ is very intricate and difficult to determine.

In this paper we give generators of the Jacquet module of a principal series representation generated by the trivial K-type. Let $\mathbb{Z}$ be the ring of integers, $\mathfrak{g}_0$ the Lie algebra of G, θ a Cartan involution of $\mathfrak{g}_0$, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ the Iwasawa decomposition of $\mathfrak{g}_0$, $\mathfrak{m}_0$ the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$, W the little Weyl group for $(\mathfrak{g}_0, \mathfrak{a}_0)$, $e \in W$ the unit element of W, Σ the restricted root system for $(\mathfrak{g}_0, \mathfrak{a}_0)$, $\mathfrak{g}_{0,\alpha}$ the root space for $\alpha \in \Sigma$, Σ^+ the positive system of Σ such that $\mathfrak{n}_0 = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{0,\alpha}$, $\rho = \sum_{\alpha\in\Sigma^+}(\dim \mathfrak{g}_{0,\alpha}/2)\alpha$, $\mathcal{P} = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z}\}$, $\mathcal{P}^+ = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z},\ n_\alpha \geq 0\}$ and $U(\lambda)$ the principal series representation with an infinitesimal character λ generated by the trivial K-type. In this paper we prove the following theorem.

Theorem 1.1 (Theorem 3.9, Theorem 4.1). *Assume that λ is regular. Set $\mathcal{W}(w) = \{w' \in W \mid w\lambda - w'\lambda \in 2\mathcal{P}^+\}$ for $w \in W$. Then there exist generators $\{v_w \mid w \in W\}$ of $J(U(\lambda))$ such that*

$$\begin{cases} (H - (\rho + w\lambda))v_w \in \sum_{w'\in\mathcal{W}(w)} U(\mathfrak{g})v_{w'} \text{ for all } H \in \mathfrak{a}_0, \\ Xv_w \in \sum_{w'\in\mathcal{W}(w)} U(\mathfrak{g})v_{w'} \text{ for all } X \in \mathfrak{m}_0 \oplus \theta(\mathfrak{n}_0). \end{cases}$$

Graduate School of Mathematical Sciences, the University of Tokyo, 3–8–1 Komaba, Meguro-ku, Tokyo 153–8914, Japan.
E-mail address: abenori@ms.u-tokyo.ac.jp
2000 *Mathematics Subject Classification*. 22E47.

We enumerate $W = \{w_1, w_2, \dots, w_r\}$ such that $\operatorname{Re} w_1\lambda \ge \operatorname{Re} w_2\lambda \ge \cdots \ge \operatorname{Re} w_r\lambda$. Set $V_i = \sum_{j\ge i} U(\mathfrak{g})v_{w_j}$. Then by Theorem 1.1 we have the surjective map $M(w_i\lambda) \to V_i/V_{i+1}$ where $M(w_i\lambda)$ is a generalized Verma module (See Definition 4.4). This map is isomorphic. Namely we can prove the following theorem.

Theorem 1.2 (Theorem 4.5). *There exists a filtration $J(U(\lambda)) = V_1 \supset V_2 \supset \cdots \supset V_{r+1} = 0$ of $J(U(\lambda))$ such that $V_i/V_{i+1} \simeq M(w_i\lambda)$. Moreover if $w\lambda - \lambda \notin 2\mathcal{P}$ for $w \in W \setminus \{e\}$ then $J(U(\lambda)) \simeq \bigoplus_{w\in W} M(w\lambda)$.*

This theorem does not need the assumption that λ is regular. In the case of G is split and $U(\lambda)$ is irreducible, Collingwood [Col91] proved Theorem 1.2.

For example, we obtain the following in case of $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$: Choose a basis $\{H, E_+, E_-\}$ of $\mathfrak{g}_0$ such that $\mathbb{R}H = \mathfrak{a}_0$, $\mathbb{R}E_+ = \mathfrak{n}_0$, $[H, E_\pm] = \pm 2E_\pm$ and $E_- = \theta(E_+)$. Then $\Sigma^+ = \{2\alpha\}$ where $\alpha(H) = 1$. Let $\lambda = r\alpha$ for $r \in \mathbb{C}$. We may assume $\operatorname{Re} r \ge 0$. By Theorems 1.1 and 1.2, we have the exact sequence

$$0 \longrightarrow M(-r\alpha) \longrightarrow J(U(r\alpha)) \longrightarrow M(r\alpha) \longrightarrow 0.$$

Consider the case λ is integral, i.e., $2r \in \mathbb{Z}$. If $r \notin \mathbb{Z}$ then this sequence splits by Theorem 1.2. On the other hand, if $r \in \mathbb{Z}$ then by the direct calculation using the method introduced in this paper we can show it does not split. Notice that $U(r\alpha)$ is irreducible if and only if $r \in \mathbb{Z}$. Then we have the following; if λ is integral then $J(U(\lambda))$ is isomorphic to the direct sum of generalized Verma modules if and only if $U(\lambda)$ is reducible.

Our method is based on the paper of Kashiwara and Oshima [KO77]. In Section 2 we show fundamental properties of Jacquet modules and introduce a certain extension of the universal enveloping algebra. An analog of the theory of Kashiwara and Oshima is established in Section 3. In Section 4 we prove our main theorem in the case of a regular infinitesimal character using the result of Section 3. We complete the proof in Section 5 using the translation principle.

Acknowledgments

The author is grateful to his advisor Hisayosi Matumoto for his advice and support. He would also like to thank Professor Toshio Oshima for his comments.

Notations

Throughout this paper we use the following notations. As usual we denote the ring of integers, the set of non-negative integers, the set of positive integers, the real number field and the complex number field by $\mathbb{Z}, \mathbb{Z}_{\ge 0}, \mathbb{Z}_{>0}, \mathbb{R}$ and $\mathbb{C}$ respectively. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. Fix a Cartan involution θ of $\mathfrak{g}_0$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ be the decomposition of $\mathfrak{g}_0$ into the $+1$ and -1 eigenspaces for θ. Take a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{s}_0$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ be the corresponding Iwasawa decomposition of $\mathfrak{g}_0$. Set $\mathfrak{m}_0 = \{X \in \mathfrak{k}_0 \mid [H, X] = 0 \text{ for all } H \in \mathfrak{a}_0\}$. Then $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is a minimal parabolic subalgebra of $\mathfrak{g}_0$. Write $\mathfrak{g}$ for the complexification of $\mathfrak{g}_0$ and $U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$. We apply analogous notations to other Lie algebras.

Set $\mathfrak{a}^* = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{a}, \mathbb{C})$ and $\mathfrak{a}_0^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{R})$. Let $\Sigma \subset \mathfrak{a}^*$ be the restricted root system for $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}_\alpha$ the root space for $\alpha \in \Sigma$. Let Σ^+ be the positive root system determined by $\mathfrak{n}$, i.e.,

$\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$. Σ^+ determines the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$. We define the total order on $\mathfrak{a}_0^*$ by the following; for $c_i, d_i \in \mathbb{R}$ we define $\sum_i c_i\alpha_i > \sum_i d_i\alpha_i$ if and only if there exists an integer k such that $c_1 = d_1, \dots, c_k = d_k$ and $c_{k+1} > d_{k+1}$. Let $\{H_1, H_2, \dots, H_l\}$ be the dual basis of $\{\alpha_i\}$. Write W for the little Weyl group for $(\mathfrak{g}_0, \mathfrak{a}_0)$ and e for the unit element of W. Set $\mathcal{P} = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z}\}$, $\mathcal{P}^+ = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z}_{\geq 0}\}$ and $\mathcal{P}^{++} = \mathcal{P}^+ \setminus \{0\}$. Let m be a dimension of $\mathfrak{n}$. Fix a basis $E_1, E_2, \dots, E_m$ of $\mathfrak{n}$ such that each E_i is a restricted root vector. Let β_i be a restricted root vector such that $E_i \in \mathfrak{g}_{\beta_i}$. For $\mathbf{n} = (\mathbf{n}_i) \in \mathbb{Z}_{\geq 0}^m$ we denote $E_1^{\mathbf{n}_1}E_2^{\mathbf{n}_2}\cdots E_m^{\mathbf{n}_m}$ by $E^{\mathbf{n}}$.

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_{\geq 0}^n$, we write $|x| = x_1 + x_2 + \cdots + x_n$ and $x! = x_1!\,x_2!\cdots x_n!$.

For a $\mathbb{C}$-algebra R, let $M(r, r', R)$ be the space of $r \times r'$ matrices with entries in R and $M(r, R) = M(r, r, R)$. Write $1_r \in M(r, R)$ for the identity matrix.

§2. Jacquet modules and fundamental properties

Definition 2.1 (Jacquet module). *Let U be a $U(\mathfrak{g})$-module. Define modules $\widehat{J}(U)$ and $J(U)$ by*

$$\widehat{J}(U) = \varprojlim_k U/\mathfrak{n}^k U,$$
$$J(U) = \widehat{J}(U)_{\mathfrak{a}\text{-finite}} = \{u \in \widehat{J}(U) \mid \dim U(\mathfrak{a})u < \infty\}.$$

We call $J(U)$ the Jacquet module of U.

Set $\widehat{\mathcal{E}}(\mathfrak{n}) = \varprojlim_k U(\mathfrak{n})/\mathfrak{n}^k U(\mathfrak{n})$.

Proposition 2.2. *(1) The $\mathbb{C}$-algebra $\widehat{\mathcal{E}}(\mathfrak{n})$ is right and left Noetherian.*

(2) The $\mathbb{C}$-algebra $\widehat{\mathcal{E}}(\mathfrak{n})$ is flat over $U(\mathfrak{n})$.

(3) If U is a finitely generated $U(\mathfrak{n})$-module then $\varprojlim_k U/\mathfrak{n}^k U = \widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} U$.

(4) Let $S = (S_k)$ be an element of $M(r, \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))$ and $(a_n) \in \mathbb{C}^{\mathbb{Z}_{\geq 0}}$. Define $\sum_{n=0}^\infty a_n S^n = (\sum_{n=0}^k a_n S_k^n)_k$. Then $\sum_{n=0}^\infty a_n S^n \in M(r, \widehat{\mathcal{E}}(\mathfrak{n}))$.

Proof. Since Stafford and Wallach [SW82, Theorem 2.1] show that $\mathfrak{n}U(\mathfrak{n}) \subset U(\mathfrak{n})$ satisfies the Artin-Rees property, the usual argument of the proof for commutative rings can be applicable to prove (1), (2) and (3). (4) is obvious. □

Corollary 2.3. *Let S be an element of $M(r, \widehat{\mathcal{E}}(\mathfrak{n}))$ such that $S - 1_r \in M(r, \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))$. Then S is invertible.*

Proof. Set $T = 1_r - S$. By Proposition 2.2, $R = \sum_{n=0}^\infty T^n \in M(r, \widehat{\mathcal{E}}(\mathfrak{n}))$. Then $SR = RS = 1_r$. □

We can prove the following proposition in a similar way to that of Goodman and Wallach [GW80, Lemma 2.2]. For the sake of completeness we give a proof.

Proposition 2.4. *Let U be a $U(\mathfrak{a}\oplus\mathfrak{n})$-module which is finitely generated as a $U(\mathfrak{n})$-module. Assume that every element of U is $\mathfrak{a}$-finite. For $\mu\in\mathfrak{a}^*$ set*

$$U_\mu = \{u\in U \mid \text{For all } H\in\mathfrak{a} \text{ there exists a positive integer } N \text{ such that } (H-\mu(H))^N u = 0\}.$$

Then

$$\widehat{J}(U)\simeq\prod_{\mu\in\mathfrak{a}^*}U_\mu.$$

Proof. For $k\in\mathbb{Z}_{>0}$ put $S_k=\{\mu\in\mathfrak{a}^*\mid U_\mu\neq 0,\ U_\mu\not\subset\mathfrak{n}^kU\}$. Since U is finitely generated, $\dim U/\mathfrak{n}^kU<\infty$. Therefore S_k is a finite set. Define a map $\varphi\colon\prod_{\mu\in\mathfrak{a}^*}U_\mu\to\widehat{J}(U)$ by

$$\varphi((x_\mu)_{\mu\in\mathfrak{a}^*})=\left(\sum_{\mu\in S_k}x_\mu \pmod{\mathfrak{n}^kU}\right)_k.$$

First we show that φ is injective. Assume $\varphi((x_\mu)_{\mu\in\mathfrak{a}^*})=0$. We have $\sum_{\mu\in S_k}x_\mu\in\mathfrak{n}^kU$ for all $k\in\mathbb{Z}_{>0}$. Since $\mathfrak{n}^kU$ is $\mathfrak{a}$-stable and S_k is a finite set, $x_\mu\in\mathfrak{n}^kU$ for all $\mu\in\mathfrak{a}^*$, thus we have $x_\mu=0$.

We have to show that φ is surjective. Let $x=(x_k \pmod{\mathfrak{n}^kU})_k$ be an element of $\widehat{J}(U)$. Since every element of U is $\mathfrak{a}$-finite, we have $U=\bigoplus_{\mu\in\mathfrak{a}^*}U_\mu$. Let $p_\mu\colon U\to U_\mu$ be the projection. The $U(\mathfrak{n})$-module U is finitely generated and therefore for all $\mu\in\mathfrak{a}^*$ there exists a positive integer k_μ such that $\mathfrak{n}^{k_\mu}U\cap U_\mu=0$. Notice that if $i,i'>k_\mu$ then $p_\mu(x_i)=p_\mu(x_{i'})$. Hence we have $\varphi((p_\mu(x_{k_\mu}))_{\mu\in\mathfrak{a}^*})=x$. □

We define an $(\mathfrak{a}\oplus\mathfrak{n})$-representation structure of $U(\mathfrak{n})$ by $(H+X)(u)=Hu-uH+Xu$ for $H\in\mathfrak{a}$, $X\in\mathfrak{n}$, $u\in U(\mathfrak{n})$. Then $U(\mathfrak{n})$ is a $U(\mathfrak{a}\oplus\mathfrak{n})$-module. By Proposition 2.4 $\widehat{\mathcal{E}}(\mathfrak{n})=\prod_{\mu\in\mathfrak{a}^*}U(\mathfrak{n})_\mu$. The following results are corollaries of Proposition 2.4.

Corollary 2.5. *A linear map*

$$\begin{array}{ccc}\mathbb{C}[[X_1,X_2,\dots,X_m]] & \longrightarrow & \widehat{\mathcal{E}}(\mathfrak{n})\\ \sum_{\mathbf{n}\in\mathbb{Z}^m_{\geq 0}}a_{\mathbf{n}}X^{\mathbf{n}} & \longmapsto & \left(\sum_{|\mathbf{n}|\leq k}a_{\mathbf{n}}E^{\mathbf{n}} \pmod{\mathfrak{n}^kU(\mathfrak{n})}\right)_k\end{array}$$

is bijective, where $X^{\mathbf{n}}=X_1^{\mathbf{n}_1}X_2^{\mathbf{n}_2}\cdots X_m^{\mathbf{n}_m}$ for $\mathbf{n}=(\mathbf{n}_1,\mathbf{n}_2,\dots,\mathbf{n}_m)\in\mathbb{Z}^m_{\geq 0}$.

Proof. By the Poincaré-Birkhoff-Witt theorem $\{E^{\mathbf{n}}\mid\sum_i\mathbf{n}_i\beta_i=\mu\}$ is a basis of $U(\mathfrak{n})_\mu$. This implies the corollary since $\widehat{\mathcal{E}}(\mathfrak{n})=\prod_{\mu\in\mathfrak{a}^*}U(\mathfrak{n})_\mu$. □

We denote the image of $\sum_{\mathbf{n}\in\mathbb{Z}^m_{\geq 0}}a_{\mathbf{n}}X^{\mathbf{n}}$ under the map in Corollary 2.5 by $\sum_{\mathbf{n}\in\mathbb{Z}^m_{\geq 0}}a_{\mathbf{n}}E^{\mathbf{n}}$.

Corollary 2.6. *Let U be a $U(\mathfrak{g})$-module which is finitely generated as a $U(\mathfrak{n})$-module. Assume that all elements are $\mathfrak{a}$-finite. Then $J(U)=U$.*

Proof. This follows from the following equation.

$$J(U)=\widehat{J}(U)_{\mathfrak{a}\text{-finite}}=\left(\prod_{\mu\in\mathfrak{a}^*}U_\mu\right)_{\mathfrak{a}\text{-finite}}=\bigoplus_{\mu\in\mathfrak{a}^*}U_\mu=U.$$

□

Put $\widehat{\mathscr{E}}(\mathfrak{g},\mathfrak{n}) = \widehat{\mathscr{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} U(\mathfrak{g})$. We can define a $\mathbb{C}$-algebra structure of $\widehat{\mathscr{E}}(\mathfrak{g},\mathfrak{n})$ by

$$\begin{aligned}
(f\otimes 1)(1\otimes u) &= f\otimes u,\\
(1\otimes u)(1\otimes u') &= 1\otimes (uu'),\\
(f\otimes 1)(f'\otimes 1) &= (ff')\otimes 1,\\
(1\otimes X)(f\otimes 1) &= \sum_{\mathbf{n}\in\mathbb{Z}^r_{\geq 0}} \frac{1}{\mathbf{n}!}\frac{\partial^{|\mathbf{n}|}}{\partial E^{\mathbf{n}}} f\otimes ((\mathrm{ad}(E))^{\mathbf{n}})'(X),
\end{aligned}$$

where $u,u'\in U(\mathfrak{g})$, $X\in\mathfrak{g}$, $f,f'\in\widehat{\mathscr{E}}(\mathfrak{n})$, $((\mathrm{ad}(E))^{\mathbf{n}})' = (-\mathrm{ad}(E_m))^{\mathbf{n}_m}\cdots(-\mathrm{ad}(E_1))^{\mathbf{n}_1}$ and

$$\frac{\partial^{|\mathbf{n}|}}{\partial E^{\mathbf{n}}}\left(\sum_{\mathbf{m}\in\mathbb{Z}^m_{\geq 0}} a_{\mathbf{m}} E^{\mathbf{m}}\right) = \sum_{\mathbf{m}\in\mathbb{Z}^m_{\geq 0}} a_{\mathbf{m}}\frac{\mathbf{m}!}{(\mathbf{m}-\mathbf{n})!}E^{\mathbf{m}-\mathbf{n}}.$$

Notice that $\widehat{\mathscr{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})} U \simeq \widehat{\mathscr{E}}(\mathfrak{n})\otimes_{U(\mathfrak{n})} U$ as an $\widehat{\mathscr{E}}(\mathfrak{n})$-module for a $U(\mathfrak{g})$-module U. By Proposition 2.2, $\widehat{\mathscr{E}}(\mathfrak{g},\mathfrak{n})$ is flat over $U(\mathfrak{g})$. Notice that if $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ which contains $\mathfrak{n}$ then $\widehat{\mathscr{E}}(\mathfrak{n})\otimes_{U(\mathfrak{n})} U(\mathfrak{b})$ is a subalgebra of $\widehat{\mathscr{E}}(\mathfrak{g},\mathfrak{n})$. Put $\widehat{\mathscr{E}}(\mathfrak{b},\mathfrak{n}) = \widehat{\mathscr{E}}(\mathfrak{n})\otimes_{U(\mathfrak{n})} U(\mathfrak{b})$.

Let U be a $U(\mathfrak{a}\oplus\mathfrak{n})$-module such that $U = \bigoplus_{\mu\in\mathfrak{a}^*} U_\mu$. Set

$$V = \left\{ (u_\mu)_\mu \in \prod_{\mu\in\mathfrak{a}^*} U_\mu \;\middle|\; \text{there exists an element } \nu\in\mathfrak{a}_0^* \text{ such that } u_\mu = 0 \text{ for } \operatorname{Re}\mu < \nu \right\}.$$

Then we can define an $\mathfrak{a}$-module homomorphism

$$\varphi\colon \widehat{\mathscr{E}}(\mathfrak{a}\oplus\mathfrak{n},\mathfrak{n})\otimes_{U(\mathfrak{a}\oplus\mathfrak{n})} U \simeq \left(\prod_{\mu\in\mathfrak{a}^*} U(\mathfrak{n})_\mu\right)\otimes_{U(\mathfrak{n})}\left(\bigoplus_{\mu'\in\mathfrak{a}^*} U_{\mu'}\right) \to V$$

by $\varphi((f_\mu)_{\mu\in\mathfrak{a}^*}\otimes(u_{\mu'})_{\mu'\in\mathfrak{a}^*}) = (\sum_{\mu+\mu'=\lambda} f_\mu u_{\mu'})_{\lambda\in\mathfrak{a}^*}$. Notice that the composition $U\to\widehat{U}\to V$ is equal to the inclusion map $U\hookrightarrow V$.

We consider the case $U = U(\mathfrak{g})$. Define an $(\mathfrak{a}\oplus\mathfrak{n})$-module structure of $U(\mathfrak{g})$ by $(H+X)(u) = Hu - uH + Xu$ for $H\in\mathfrak{a}$, $X\in\mathfrak{n}$, $u\in U(\mathfrak{g})$. We have a map

$$\begin{aligned}
&\varphi\colon \widehat{\mathscr{E}}(\mathfrak{g},\mathfrak{n}) \to\\
&\quad\left\{ (P_\mu)_{\mu\in\mathfrak{a}^*} \in \prod_{\mu\in\mathfrak{a}^*} U(\mathfrak{g})_\mu \;\middle|\; \text{there exists an element } \nu\in\mathfrak{a}_0^* \text{ such that } P_\mu = 0 \text{ for } \operatorname{Re}\mu<\nu \right\}.
\end{aligned}$$

We write $\varphi(P) = (P^{(\mu)})_{\mu\in\mathfrak{a}^*}$. Put $P^{(H,z)} = \sum_{\mu(H)=z} P^{(\mu)}$ for $z\in\mathbb{C}$ and $H\in\mathfrak{a}$ such that $\operatorname{Re}\alpha(H) > 0$ for all $\alpha\in\Sigma^+$.

By the definition we have the following proposition.

Proposition 2.7. *(1) Assume that U is finitely generated as a $U(\mathfrak{n})$-module. Let $\varphi\colon \widehat{\mathscr{E}}(\mathfrak{a}\oplus\mathfrak{n},\mathfrak{n})\otimes_{U(\mathfrak{n})} U \to \prod_{\mu\in\mathfrak{a}^*} U_\mu$ be an $\mathfrak{a}$-module homomorphism defined as above. Then φ is coincide with the map given in Proposition 2.4. In particular φ is isomorphic.*

(2) We have $(PQ)^{(\lambda)} = \sum_{\mu+\mu'=\lambda} P^{(\mu)}Q^{(\mu')}$ *for* $P, Q \in \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ *and* $\lambda \in \mathfrak{a}^*$.

(3) We have

$$\left(\sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}^m} a_{\mathbf{n}} E^{\mathbf{n}}\right)^{(\lambda)} = \sum_{\sum_i \mathbf{n}_i\beta_i=\lambda} a_{\mathbf{n}} E^{\mathbf{n}}$$

for $\lambda \in \mathfrak{a}^*$.

Proposition 2.8. *Let* U *be a* $U(\mathfrak{g})$*-module which is finitely generated as a* $U(\mathfrak{n})$*-module. We take generators* $v_1, v_2, \dots, v_n$ *of an* $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$*-module* $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U$ *and set* $V = \sum_i U(\mathfrak{g})v_i \subset \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U$*. Define the surjective map* $\psi\colon \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V \to \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U$ *by* $\psi(f \otimes v) = fv$*. Assume that there exist weights* $\lambda_i \in \mathfrak{a}^*$ *and a positive integer* N *such that* $(H - \lambda_i(H))^N v_i = 0$ *for all* $H \in \mathfrak{a}$ *and* $1 \leq i \leq n$*. Let* $\varphi\colon \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V \to \prod_{\mu\in\mathfrak{a}^*} V_\mu$ *be the map defined as above. Then there exists a unique map* $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U \to \prod_{\mu\in\mathfrak{a}^*} V_\mu$ *such that the diagram*

$$\begin{tikzcd} \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V \arrow[r, "\varphi"] \arrow[d, "\psi"] & \prod_{\mu\in\mathfrak{a}^*} V_\mu \\ \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U \arrow[ur, dotted] & \end{tikzcd}$$

is commutative.

Proof. Set $\widehat{U} = \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U$ and $\widehat{V} = \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V$. Take $f^{(i)} \in \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ and $v^{(i)} \in V$ such that $\psi(\sum_i f^{(i)} \otimes v^{(i)}) = 0$. We have to show $\varphi(\sum_i f^{(i)} \otimes v^{(i)}) = 0$. Since $\widehat{V} = \widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} V$, we may assume $f_i \in \widehat{\mathcal{E}}(\mathfrak{n})$. We can write $f^{(i)} = (f^{(i)}_\mu)_{\mu\in\mathfrak{a}^*}$ by the isomorphism $\widehat{\mathcal{E}}(\mathfrak{n}) \simeq \prod_{\mu\in\mathfrak{a}^*} U(\mathfrak{n})_\mu$. Since $V = \bigoplus_{\mu'\in\mathfrak{a}^*} V_{\mu'}$, we can write $v_i = \sum_{\mu'\in\mathfrak{a}^*} v^{(i)}_{\mu'}$, $v^{(i)}_{\mu'} \in V_{\mu'}$. We have to show $\sum_i \sum_{\mu+\mu'=\lambda} f^{(i)}_\mu v^{(i)}_{\mu'} = 0$ for all $\lambda \in \mathfrak{a}^*$. Since U is a finitely generated $U(\mathfrak{n})$-module we have $\widehat{U} = \varprojlim_k U/\mathfrak{n}^k U = \varprojlim_k \widehat{U}/\mathfrak{n}^k\widehat{U}$. It is sufficient to prove $\sum_i \sum_{\mu+\mu'=\lambda} f^{(i)}_\mu v^{(i)}_{\mu'} \in \mathfrak{n}^k\widehat{U}$ for all $k \in \mathbb{Z}_{>0}$.

Fix $\lambda \in \mathfrak{a}^*$ and $k \in \mathbb{Z}_{>0}$. We can choose an element $\nu \in \mathfrak{a}_0^*$ such that $\bigoplus_{\operatorname{Re}\mu\geq\nu} U(\mathfrak{n})_\mu \subset \mathfrak{n}^k U(\mathfrak{n})$. Then $0 = \varphi(\sum_i f^{(i)} \otimes v^{(i)}) \equiv \sum_i \sum_{\operatorname{Re}\mu<\nu} f^{(i)}_\mu v^{(i)}_{\mu'} \pmod{\mathfrak{n}^k\widehat{U}}$. Notice that following two sets are finite.

$$\{\mu \mid \operatorname{Re}(\mu) < \nu \text{ and there exists an integer } i \text{ such that } f^{(i)}_\mu \neq 0\},$$
$$\{\mu' \mid \text{there exists an integer } i \text{ such that } v^{(i)}_{\mu'} \neq 0\}.$$

This implies $\sum_i \sum_{\mu+\mu'=\lambda} f^{(i)}_\mu v^{(i)}_{\mu'} \in \mathfrak{n}^k\widehat{U}$. □

The following result is a corollary of Proposition 2.8.

Corollary 2.9. *In the setting of Proposition 2.8, we have the following. Let* P_i $(1 \leq i \leq n)$ *be elements of* $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ *such that* $\sum_{i=1}^n P_i v_i = 0$*. Then* $\sum_i P_i^{(\lambda-\lambda_i)} v_i = 0$ *for all* $\lambda \in \mathfrak{a}^*$.

§3. Construction of special elements

Let Λ be a subset of $\mathcal{P}$. Put $\Lambda^+ = \Lambda \cap \mathcal{P}^+$ and $\Lambda^{++} = \Lambda \cap \mathcal{P}^{++}$. We define vector spaces $U(\mathfrak{g})_\Lambda, U(\mathfrak{n})_\Lambda, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda$ and $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})_\Lambda$ by

$$\begin{aligned} U(\mathfrak{g})_\Lambda &= \{P \in U(\mathfrak{g}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\},\\ U(\mathfrak{n})_\Lambda &= \{P \in U(\mathfrak{n}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\},\\ \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda &= \{P \in \widehat{\mathcal{E}}(\mathfrak{n}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\},\\ \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})_\Lambda &= \{P \in \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\}. \end{aligned}$$

Put $(\mathfrak{n}U(\mathfrak{n}))_\Lambda = \mathfrak{n}U(\mathfrak{n}) \cap U(\mathfrak{n})_\Lambda$ and $(\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda = \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}) \cap \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda$.

We assume that Λ is a subgroup of $\mathfrak{a}^*$. Then $U(\mathfrak{g})_\Lambda, U(\mathfrak{n})_\Lambda, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda$ and $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})_\Lambda$ are $\mathbb{C}$-algebras. Let U be a $U(\mathfrak{g})_\Lambda$-module which is finitely generated as a $U(\mathfrak{n})_\Lambda$-module. Let $u_1, u_2, \dots, u_N$ be generators of U as a $U(\mathfrak{n})_\Lambda$-module. Put $u = {}^t(u_1, u_2, \dots, u_N)$, $\overline{U} = U/(\mathfrak{n}U(\mathfrak{n}))_\Lambda U$ and $\overline{u} = u \pmod{(\mathfrak{n}U(\mathfrak{n}))_\Lambda}$. The module $\overline{U}$ has an $\mathfrak{a}$-module structure induced from that of U. By the assumption we have $\dim \overline{U} < \infty$. Let $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathfrak{a}^*$ ($\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_2 \geq \cdots \geq \operatorname{Re}\lambda_r$) be eigenvalues of $\mathfrak{a}$ on $\overline{U}$ with multiplicities. We can choose a basis $\overline{v_1}, \overline{v_2}, \dots, \overline{v_r}$ of $\overline{U}$ and a linear map $\overline{Q}\colon \mathfrak{a} \to M(r,\mathbb{C})$ such that

$$\begin{cases} H\overline{v} = \overline{Q}(H)\overline{v} \text{ for all } H \in \mathfrak{a},\\ \overline{Q}(H)_{ii} = \lambda_i(H) \text{ for all } H \in \mathfrak{a},\\ \text{if } i > j \text{ then } \overline{Q}(H)_{ij} = 0 \text{ for all } H \in \mathfrak{a},\\ \text{if } \lambda_i \neq \lambda_j \text{ then } \overline{Q}(H)_{ij} = 0 \text{ for all } H \in \mathfrak{a}, \end{cases}$$

where $\overline{v} = {}^t(\overline{v_1}, \overline{v_2}, \dots, \overline{v_r})$. Take $\overline{A} \in M(N, r, \mathbb{C})$ and $\overline{B} \in M(r, N, \mathbb{C})$ such that $\overline{v} = \overline{B}\overline{u}$ and $\overline{u} = \overline{A}\overline{v}$.

Set $\widehat{U} = \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})_\Lambda \otimes_{U(\mathfrak{g})_\Lambda} U$.

Theorem 3.1. *There exist matrices $A \in M(N, r, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ and $B \in M(r, N, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ such that the following conditions hold:*

- *There exists a linear map $Q\colon \mathfrak{a} \to M(r, U(\mathfrak{n})_\Lambda)$ such that*

$$\begin{cases} Hv = Q(H)v \text{ for all } H \in \mathfrak{a},\\ Q(H) - \overline{Q}(H) \in M(r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda) \text{ for all } H \in \mathfrak{a},\\ \text{if } \lambda_i - \lambda_j \notin \Lambda^+ \text{ then } Q(H)_{ij} = 0 \text{ for all } H \in \mathfrak{a},\\ \text{if } \lambda_i - \lambda_j \in \Lambda^+ \text{ then } [H', Q(H)_{ij}] = (\lambda_i - \lambda_j)(H')Q(H)_{ij} \text{ for all } H, H' \in \mathfrak{a}, \end{cases}$$

  where $v = Bu$.

- *We have $u = ABu$.*

- *We have $A - \overline{A} \in M(N, r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$ and $B - \overline{B} \in M(r, N, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$.*

For the proof we need some lemmas. Put $w = \overline{B}u \in \widehat{U}^r$.

Lemma 3.2. *For $H \in \mathfrak{a}$ there exists a matrix $R \in M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$ such that $Hw = (\overline{Q}(H)+R)w$ in $\widehat{U}^r$.*

Proof. Since $w \pmod{((\mathfrak{n}U(\mathfrak{n}))_\Lambda U)^r} = \overline{v}$, we have $Hw - \overline{Q}(H)w \in ((\mathfrak{n}U(\mathfrak{n}))_\Lambda U)^r$. Hence there exists a matrix $R_1 \in M(N, r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ such that $Hw - \overline{Q}(H)w = R_1 u$. Similarly we can choose a matrix $S \in M(N, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ which satisfies $u = \overline{A}w + Su$. Put $R = R_1(1-S)^{-1}\overline{A}$. Then $(H - \overline{Q}(H) - R)w = R_1 u - R_1(1-S)^{-1}\overline{A}w = 0$. □

Lemma 3.3. *Let $\lambda \in \mathbb{C}$ and $Q_0, R \in M(r, \mathbb{C})$. Assume that Q_0 is an upper triangular matrix. Then there exist matrices $L, T \in M(r, \mathbb{C})$ such that*

$$\begin{cases} \lambda L - [Q_0, L] = T + R, \\ \textit{if } (Q_0)_{ii} - (Q_0)_{jj} \neq \lambda \textit{ then } T_{ij} = 0. \end{cases}$$

Proof. Since $(Q_0)_{ij} = 0$ for $i > j$, we have

$$\begin{aligned} (\lambda L - [Q_0, L])_{ij} &= \lambda L_{ij} - \sum_{k=1}^{r} ((Q_0)_{ik} L_{kj} - L_{ik}(Q_0)_{kj}) \\ &= \lambda L_{ij} - \sum_{k=i}^{r} (Q_0)_{ik} L_{kj} + \sum_{k=1}^{j} L_{ik}(Q_0)_{kj} \\ &= (\lambda - ((Q_0)_{ii} - (Q_0)_{jj}))L_{ij} - \sum_{k=i+1}^{r} (Q_0)_{ik} L_{kj} + \sum_{k=1}^{j-1} L_{ik}(Q_0)_{kj}. \end{aligned}$$

Hence we can choose L_{ij} and T_{ij} inductively on $(j-i)$. □

Lemma 3.4. *Let H be an element of $\mathfrak{a}$ such that $\alpha(H) > 0$ for all $\alpha \in \Sigma^+$. Let $Q_0 \in M(r, \mathbb{C})$, $R \in M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$. Assume $(Q_0)_{ij} = 0$ for $i > j$. Set $\mathcal{L}^{++} = \{\lambda(H) \mid \lambda \in \Lambda^{++}\}$. Then there exist matrices $L \in M(r, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ and $T \in M(r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ such that*

$$\begin{cases} L \equiv 1_r \pmod{(\mathfrak{n}U(\mathfrak{n}))_\Lambda}, \\ (H1_r - Q_0 - T)L = L(H1_r - Q_0 - R), \\ \textit{if } (Q_0)_{ii} - (Q_0)_{jj} \notin \mathcal{L}^{++} \textit{ then } T_{ij} = 0, \\ \textit{if } (Q_0)_{ii} - (Q_0)_{jj} \in \mathcal{L}^{++} \textit{ then } [H, T_{ij}] = ((Q_0)_{ii} - (Q_0)_{jj})T_{ij}. \end{cases}$$

Proof. Set $\mathcal{L} = \{\lambda(H) \mid \lambda \in \Lambda\}$ and $\mathcal{L}^+ = \{\lambda(H) \mid \lambda \in \Lambda^+\}$. Put $f(\mathbf{n}) = \sum_i \mathbf{n}_i \beta_i$ for $\mathbf{n} = (\mathbf{n}_i) \in \mathbb{Z}^m$. Set $\widetilde{\Lambda} = \{\mathbf{n} \in \mathbb{Z}^m_{\geq 0} \mid f(\mathbf{n}) \in \Lambda\}$. We define the order on $\widetilde{\Lambda}$ by $\mathbf{n} < \mathbf{n}'$ if and only if $f(\mathbf{n}) < f(\mathbf{n}')$.

By Corollary 2.5, we can write $R = \sum_{\mathbf{n} \in \widetilde{\Lambda}} R_{\mathbf{n}} E^{\mathbf{n}}$ where $R_{\mathbf{n}} \in M(r, \mathbb{C})$. We have $R_{\mathbf{0}} = 0$ where $\mathbf{0} = (0)_i \in \widetilde{\Lambda}$ since $R \in M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$. We have to show the existence of L and T. Write $L = 1_r + \sum_{\mathbf{n} \in \widetilde{\Lambda}} L_{\mathbf{n}} E^{\mathbf{n}}$ and $T = \sum_{\mathbf{n} \in \widetilde{\Lambda}} T_{\mathbf{n}} E^{\mathbf{n}}$. Then $(H1_r - Q_0 - T)L = L(H1_r - Q_0 - R)$ is equivalent to

$$f(\mathbf{n})(H)L_{\mathbf{n}} - [Q_0, L_{\mathbf{n}}] = T_{\mathbf{n}} + S_{\mathbf{n}} - R_{\mathbf{n}},$$

where $S_{\mathbf{n}}$ is defined by

$$\sum_{\mathbf{n}\in\widetilde{\Lambda}} S_{\mathbf{n}}E^{\mathbf{n}} = T(L-1_r)-(L-1_r)R.$$

By Proposition 2.7 the above equation is equivalent to

$$\sum_{f(\mathbf{n})=\mu} S_{\mathbf{n}}E^{\mathbf{n}} = \sum_{f(\mathbf{k})+f(\mathbf{l})=\mu} T_{\mathbf{k}}L_{\mathbf{l}}E^{\mathbf{k}}E^{\mathbf{l}} - \sum_{f(\mathbf{k})+f(\mathbf{l})=\mu} L_{\mathbf{k}}R_{\mathbf{l}}E^{\mathbf{k}}E^{\mathbf{l}}$$

for all $\mu\in\mathfrak{a}^*$. Notice that $L_{\mathbf{0}}=T_{\mathbf{0}}=0$. $S_{\mathbf{n}}$ can be defined from the data $\{T_{\mathbf{k}}\mid \mathbf{k}<\mathbf{n}\}$, $\{L_{\mathbf{k}}\mid \mathbf{k}<\mathbf{n}\}$ and $\{R_{\mathbf{k}}\mid \mathbf{k}<\mathbf{n}\}$.

Now we prove the existence of L and T. We choose the $L_{\mathbf{n}}$ and $T_{\mathbf{n}}$ which satisfy

$$\begin{cases} L_{\mathbf{0}}=0, \quad T_{\mathbf{0}}=0, \\ f(\mathbf{n})(H)L_{\mathbf{n}}-[Q_0,L_{\mathbf{n}}]=T_{\mathbf{n}}+S_{\mathbf{n}}-R_{\mathbf{n}}, \\ \text{if } (Q_0)_{ii}-(Q_0)_{jj}\neq f(\mathbf{n})(H) \text{ then } (T_{\mathbf{n}})_{ij}=0. \end{cases}$$

By Lemma 3.3, we can choose such $L_{\mathbf{n}}$ and $T_{\mathbf{n}}$ inductively. Put $L=1_r+\sum_{\mathbf{n}\in\widetilde{\Lambda}}L_{\mathbf{n}}E^{\mathbf{n}}$ and $T=\sum_{\mathbf{n}\in\widetilde{\Lambda}}T_{\mathbf{n}}E^{\mathbf{n}}$. Since

$$[H,T_{ij}]=\sum_{\mathbf{n}\in\widetilde{\Lambda}}(f(\mathbf{n})(H))(T_{\mathbf{n}})_{ij}E^{\mathbf{n}}=((Q_0)_{ii}-(Q_0)_{jj})T_{ij},$$

L and T satisfy the conditions of the lemma. □

Proof of Theorem 3.1. We can choose positive integers $C=(C_i)\in\mathbb{Z}^l_{>0}$ such that

$$\{\alpha\in\Lambda^{++}\mid \alpha(\textstyle\sum_i C_iH_i)=(\lambda_i-\lambda_j)(\sum_i C_iH_i)\} = \begin{cases}\{\lambda_i-\lambda_j\} & (\lambda_i-\lambda_j\in\Lambda^{++}), \\ \emptyset & (\lambda_i-\lambda_j\notin\Lambda^{++}).\end{cases}$$

The existence of such C is shown by Oshima [Osh84, Lemma 2.3]. Set $X=\sum_i C_iH_i$. Notice that $(\lambda_i-\lambda_j)(X)\in\{\mu(X)\mid\mu\in\Lambda^{++}\}$ if and only if $\lambda_i-\lambda_j\in\Lambda^{++}$. By Lemma 3.4, there exist $T\in M(r,(\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ and $L\in M(r,\widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ such that

$$\begin{cases} L\equiv 1_r \pmod{(\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda}, \\ (X1_r-\overline{Q}(X)-T)L=L(X1_r-\overline{Q}(X)-R), \\ \text{if } \lambda_i-\lambda_j\notin\Lambda^{++} \text{ then } T_{ij}=0, \\ \text{if } \lambda_i-\lambda_j\in\Lambda^{++} \text{ then } [X,T_{ij}]=(\lambda_i-\lambda_j)(X)T_{ij}. \end{cases}$$

Let $S\in M(N,(\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ such that $u=\overline{A}w+Su$. Put $A=(1-S)^{-1}\overline{A}L^{-1}$, $B=L\overline{B}$ and $v=(v_1,v_2,\ldots,v_r)=Bu=Lw$ then $ABu=(1-S)^{-1}\overline{A}L^{-1}L\overline{B}u=(1-S)^{-1}\overline{A}w=u$. Moreover, we have $(X1_r-\overline{Q}(X)-T)v=0$. Since $[X,T_{ij}]=(\lambda_i-\lambda_j)(X)T_{ij}$, we have $(X-\lambda_i(X))^rv_i=0$.

Fix a positive integer k such that $1\le k\le l$. We can choose a matrix $R_k\in M(r,(\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$ such that $H_kw=(\overline{Q}(H_k)+R_k)w$ by Lemma 3.2. Set $T_k=H_k1_r-\overline{Q}(H_k)-L(H_k1_r-\overline{Q}(H_k)-R_k)L^{-1}$. Then we have $(H_k1_r-\overline{Q}(H_k)-T_k)v=0$, i.e.,

$$H_kv_i-\sum_{j=1}^r\overline{Q}(H_k)_{ij}v_j-\sum_{j=1}^r(T_k)_{ij}v_j=0$$

for each $i = 1, 2, \dots, r$. By Corollary 2.9, we have

$$H_k v_i - \sum_{j=1}^{r} \overline{Q}(H_k)_{ij} v_j - \sum_{j=1}^{r} (T_k)_{ij}^{(X,(\lambda_i-\lambda_j)(X))} v_j = 0.$$

Define $T'_k \in M(r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ by $(T'_k)_{ij} = (T_k)_{ij}^{(X,(\lambda_i-\lambda_j)(X))}$. Since $(T_k)_{ij}^{(\mu)} = 0$ for $\mu \notin \Lambda^{++}$, we have

$$(T_k)_{ij}^{(X,(\lambda_i-\lambda_j)(X))} = \sum_{\mu\in\Lambda^{++},\ \mu(X)=(\lambda_i-\lambda_j)(X)} (T_k)_{ij}^{(\mu)} = \begin{cases} (T_k)_{ij}^{(\lambda_i-\lambda_j)} & (\lambda_i - \lambda_j \in \Lambda^{++}), \\ 0 & (\lambda_i - \lambda_j \notin \Lambda^{++}) \end{cases}$$

by the condition of C. In particular $[H, (T'_k)_{ij}] = (\lambda_i - \lambda_j)(H)$ for all $H \in \mathfrak{a}$. Put $Q(\sum x_i H_i) = \overline{Q}(\sum x_i H_i) + \sum x_i T'_i$ for $(x_1, x_2, \dots, x_l) \in \mathbb{C}^l$ then Q satisfies the condition of the theorem. $\square$

Set $\rho = \sum_{\alpha\in\Sigma^+}(\dim \mathfrak{g}_\alpha/2)\alpha$. From the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ we have the decomposition into the direct sum

$$U(\mathfrak{g}) = \mathfrak{n}U(\mathfrak{a} \oplus \mathfrak{n}) \oplus U(\mathfrak{a}) \oplus U(\mathfrak{g})\mathfrak{k}.$$

Let χ_1 be the projection of $U(\mathfrak{g})$ to $U(\mathfrak{a})$ with respect to this decomposition and χ_2 the algebra automorphism of $U(\mathfrak{a})$ defined by $\chi_2(H) = H - \rho(H)$ for $H \in \mathfrak{a}$. We define $\chi\colon U(\mathfrak{g})^{\mathfrak{k}} \to U(\mathfrak{a})$ by $\chi = \chi_2 \circ \chi_1$ where $U(\mathfrak{g})^{\mathfrak{k}} = \{u \in U(\mathfrak{g}) \mid Xu = uX \text{ for all } X \in \mathfrak{k}\}$. It is known that an image of $U(\mathfrak{g})^{\mathfrak{k}}$ under χ is contained in the set of W-invariant elements in $U(\mathfrak{a})$.

Fix $\lambda \in \mathfrak{a}^*$. We can define the algebra homomorphism $U(\mathfrak{a}) \to \mathbb{C}$ by $H \mapsto \lambda(H)$ for $H \in \mathfrak{a}$. We denote this map by the same letter λ. Put $\chi_\lambda = \lambda \circ \chi$. Set $U(\lambda) = U(\mathfrak{g})/(U(\mathfrak{g}) \operatorname{Ker} \chi_\lambda + U(\mathfrak{g})\mathfrak{k})$, $u_\lambda = 1 \mod (U(\mathfrak{g}) \operatorname{Ker} \chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \in U(\lambda)$ and $U(\lambda)_0 = U(\mathfrak{g})_{2\mathcal{P}} u_\lambda$. Before applying Theorem 3.1 to $U(\lambda)_0$, we give some lemmas.

Lemma 3.5. *Let $u \in U(\mathfrak{g})_\mu$ where $\mu \in \mathfrak{a}^*$. Then there exists an element $x \in U(\mathfrak{g})\mathfrak{k}$ such that $u + x \in U(\mathfrak{a} \oplus \mathfrak{n})_{\mu+2\mathcal{P}}$.*

Proof. Set $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$. We may assume $u \in U(\overline{\mathfrak{n}})_\mu$. Let $\{U_n(\overline{\mathfrak{n}})\}_{n\in\mathbb{Z}_{\geq 0}}$ be the standard filtration of $U(\mathfrak{n})$ and n the smallest integer such that $u \in U_n(\overline{\mathfrak{n}})$. We will prove the existence of x by the induction on n.

If $n = 0$ then the lemma is obvious. Assume $n > 0$. We may assume that there exist a restricted root $\alpha \in \Sigma^+$, an element $u_0 \in U_{n-1}(\overline{\mathfrak{n}})_{\mu+\alpha}$ and a vector $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $u = u_0 E_{-\alpha}$. Set $E_\alpha = \theta(E_{-\alpha})$, $u_1 = u_0 E_\alpha$, $u_2 = E_\alpha u_0$ and $u_3 = u_1 - u_2$. Then $u + u_2 + u_3 = u + u_1 \in U(\mathfrak{g})\mathfrak{k}$, $u_1, u_2 \in U(\mathfrak{g})_{\mu+2\alpha}$ and $u_3 \in U_{n-1}(\mathfrak{g})_{\mu+2\alpha}$. Using the Poincaré-Birkhoff-Witt theorem and the induction hypothesis we can choose an element $u_5 \in U(\mathfrak{a} \oplus \mathfrak{n})_{\mu+2\mathcal{P}}$ such that $u_3 - u_5 \in U(\mathfrak{g})\mathfrak{k}$. Again by the induction hypothesis we can choose an element $u_6 \in U(\mathfrak{a} \oplus \mathfrak{n})_{\mu+\alpha+2\mathcal{P}}$ such that $u_0 - u_6 \in U(\mathfrak{g})\mathfrak{k}$. Then $u + u_5 + E_\alpha u_6 \in U(\mathfrak{g})\mathfrak{k}$, $u_5 \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}$ and $E_\alpha u_6 \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\alpha+2\mathcal{P}}$. $\square$

Lemma 3.6. *The following equations hold.*

(1) $\operatorname{Ker} \chi_\lambda \subset U(\mathfrak{g})_{2\mathcal{P}}$.

(2) $U(\mathfrak{a} \oplus \mathfrak{n}) \cap (\operatorname{Ker} \chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \subset U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}} \cap (\operatorname{Ker} \chi_\lambda + U(\mathfrak{g})\mathfrak{k})$.

(3) $U(\mathfrak{a}\oplus\mathfrak{n})\cap(U(\mathfrak{a}\oplus\mathfrak{n})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})\subset U(\mathfrak{a}\oplus\mathfrak{n})(U(\mathfrak{a}\oplus\mathfrak{n})\cap(\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k}))$.

(4) $U(\mathfrak{a}\oplus\mathfrak{n})\cap(U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})=U(\mathfrak{a}\oplus\mathfrak{n})((U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})\cap U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}})$.

Proof. (1) It is sufficient to prove $U(\mathfrak{g})^{\mathfrak{k}}\subset U(\mathfrak{g})_{2\mathcal{P}}$. Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}_0$ and K its maximal compact subgroup such that $\operatorname{Lie}(K)=\mathfrak{k}_0$. Since K is connected, $U(\mathfrak{g})^{\mathfrak{k}}=U(\mathfrak{g})^K=\{u\in U(\mathfrak{g})\mid \operatorname{Ad}(k)u=u \text{ for all } k\in K\}$. Assume that G has the complexification $G_{\mathbb{C}}$. Fix a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{m}_0$. Let K_{split} and A_{split} be the analytic subgroups with Lie algebras given as the intersections of $\mathfrak{k}_0$ and $\mathfrak{a}_0$ with $[Z_{\mathfrak{g}_0}(\mathfrak{t}_0),Z_{\mathfrak{g}_0}(\mathfrak{t}_0)]$ where $Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is the centralizer of $\mathfrak{t}_0$ in $\mathfrak{g}_0$. Let F be the centralizer of A_{split} in K_{split}. Since $F\subset K$, we have $U(\mathfrak{g})^K\subset U(\mathfrak{g})^F$. On the other hand, we have $U(\mathfrak{g})^F\subset U(\mathfrak{g})_{2\mathcal{P}}$ (See Knapp [Kna02, Thorem 7.55] and Lepowsky [Lep75, Proposition 6.1, Proposition 6.4]). Hence (1) follows.

(2) Let $u\in\operatorname{Ker}\chi_\lambda$ and $x\in U(\mathfrak{g})\mathfrak{k}$ such that $u+x\in U(\mathfrak{a}\oplus\mathfrak{n})$. We can write $u=\sum_\mu u_\mu$ where $u_\mu\in U(\mathfrak{g})_\mu$. By (1), we have $u_\mu=0$ for $\mu\notin 2\mathcal{P}$. Let $\mu\in 2\mathcal{P}$. By Lemma 3.5, there exists an element $y_\mu\in U(\mathfrak{g})\mathfrak{k}$ such that $u_\mu+y_\mu\in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}=U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}$. Put $y=\sum_\mu y_\mu$. Then $u+y\in U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}$. Since $u+y\in U(\mathfrak{a}\oplus\mathfrak{n})$ and $x,y\in U(\mathfrak{g})\mathfrak{k}$ we have $y=x$ by the Poincaré-Birkhoff-Witt theorem. Hence we have $u+x=u+y\in U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}$.

(3) Let $\sum_i x_iu_i\in U(\mathfrak{a}\oplus\mathfrak{n})\operatorname{Ker}\chi_\lambda$ where $x_i\in U(\mathfrak{a}\oplus\mathfrak{n})$ and $u_i\in\operatorname{Ker}\chi_\lambda$. We write $u_i=\sum_j z_j^{(i)}v_j^{(i)}$ where $z_j^{(i)}\in U(\mathfrak{a}\oplus\mathfrak{n})$ and $v_j^{(i)}\in U(\mathfrak{k})$. Let $y\in U(\mathfrak{g})\mathfrak{k}$ and assume $\sum_i x_iu_i+y\in U(\mathfrak{a}\oplus\mathfrak{n})$. By the Poincaré-Birkhoff-Witt theorem, $\sum_i x_iu_i+y=\sum_{i,j}x_iz_j^{(i)}v_{j,0}^{(i)}$ where $v_{j,0}^{(i)}$ is the constant term of $v_j^{(i)}$. Hence $\sum_i x_iu_i+y=\sum_i x_i(u_i+\sum_j z_j^{(i)}(v_{j,0}^{(i)}-v_j^{(i)}))\in U(\mathfrak{a}\oplus\mathfrak{n})(U(\mathfrak{a}\oplus\mathfrak{n})\cap(\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k}))$.

(4) Since $\operatorname{Ker}\chi_\lambda\subset U(\mathfrak{g})^{\mathfrak{k}}$, we have

$$U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k}=U(\mathfrak{a}\oplus\mathfrak{n})(\operatorname{Ker}\chi_\lambda)U(\mathfrak{k})+U(\mathfrak{g})\mathfrak{k}=U(\mathfrak{a}\oplus\mathfrak{n})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k}.$$

By (2) and (3), we have

$$\begin{aligned}U(\mathfrak{a}\oplus\mathfrak{n})\cap(U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})&=U(\mathfrak{a}\oplus\mathfrak{n})\cap(U(\mathfrak{a}\oplus\mathfrak{n})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})\\&\subset U(\mathfrak{a}\oplus\mathfrak{n})(U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}\cap(\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k}))\\&\subset U(\mathfrak{a}\oplus\mathfrak{n})((U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})\cap U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}).\end{aligned}$$

This implies (4). □

Lemma 3.7. *We have the following equations.*

(1) $U(\lambda)_0=U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}u_\lambda$.

(2) $U(\mathfrak{n})\otimes_{U(\mathfrak{n})_{2\mathcal{P}}}U(\lambda)_0\simeq U(\lambda)$ *under the map* $p\otimes u\mapsto pu$.

Proof. (1) This is obvious from Lemma 3.5.

(2) Let $I=U(\mathfrak{a}\oplus\mathfrak{n})\cap(U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})$. We have $U(\mathfrak{a}\oplus\mathfrak{n})\otimes_{U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}}U=U(\mathfrak{n})\otimes_{U(\mathfrak{n})_{2\mathcal{P}}}U$ for any $U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}$-module U since $U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}=U(\mathfrak{a})\otimes U(\mathfrak{n})_{2\mathcal{P}}$.

By (1), we have $U(\lambda)_0 = U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}/(I\cap U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}})$. Hence

$$\begin{aligned}
U(\mathfrak{n})\otimes_{U(\mathfrak{n})_{2\mathcal{P}}} U(\lambda)_0 &= U(\mathfrak{a}\oplus\mathfrak{n})\otimes_{U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}} U(\lambda)_0\\
&= U(\mathfrak{a}\oplus\mathfrak{n})\otimes_{U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}} (U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}/(I\cap U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}))\\
&= U(\mathfrak{a}\oplus\mathfrak{n})/(U(\mathfrak{a}\oplus\mathfrak{n})(I\cap U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}))\\
&= U(\mathfrak{a}\oplus\mathfrak{n})/I\\
&= U(\lambda)
\end{aligned}$$

by Lemma 3.6 (4). $\square$

Lemma 3.8. *Let $\{U_n(\mathfrak{n})\}_{n\in\mathbb{Z}_{\geq 0}}$ be the standard filtration of $U(\mathfrak{n})$ and $U_n(\mathfrak{n})_{2\mathcal{P}} = U_n(\mathfrak{n})\cap U(\mathfrak{n})_{2\mathcal{P}}$. Set $U_{-1}(\mathfrak{n}) = U_{-1}(\mathfrak{n})_{2\mathcal{P}} = 0$, $R = \operatorname{Gr} U(\mathfrak{n})_{2\mathcal{P}} = \bigoplus_n U_n(\mathfrak{n})_{2\mathcal{P}}/U_{n-1}(\mathfrak{n})_{2\mathcal{P}}$ and $R' = \operatorname{Gr} U(\mathfrak{n}) = \bigoplus_n U_n(\mathfrak{n})/U_{n-1}(\mathfrak{n})$.*

(1) R' is a finitely generated R-module.

(2) $U(\mathfrak{n})$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module.

(3) $U(\mathfrak{n})_{2\mathcal{P}}$ is right and left Noetherian.

(4) $U(\lambda)_0$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module.

Proof. (1) Let $\Gamma = \{E^{\varepsilon} \mid \varepsilon\in\{0,1\}^m\}$. We denote the principal symbol of $u\in U(\mathfrak{n})$ by $\sigma(u)$. Notice that if $u\in U(\mathfrak{n})_{2\mathcal{P}}$ then $\sigma(u)$ is the principal symbol of u as an element of $U(\mathfrak{n})_{2\mathcal{P}}$.

We will prove that $\{\sigma(E)\mid E\in\Gamma\}$ generates R' as an R-module. Let $x\in R'$. We may assume that x is homogeneous, thus there exists an element $u\in U(\mathfrak{n})$ such that $x=\sigma(u)$. Moreover we may assume that there exist non-negative integers $p=(p_1,p_2,\dots,p_m)$ such that $u=E^p$. Choose $\varepsilon_i\in\{0,1\}$ such that $\varepsilon_i\equiv p_i \pmod 2$. Set $q_i=(p_i-\varepsilon_i)/2\in\mathbb{Z}_{\geq 0}$, $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m)$ and $q=(q_1,q_2,\dots,q_m)$. Then we have $x=\sigma(E^p)=\sigma(E^{2q})\sigma(E^{\varepsilon})$. Since $\sigma(E^{2q})\in R$, this implies that $\{\sigma(E)\mid E\in\Gamma\}$ generates R' as an R-module.

(2) This is a direct consequence of (1).

(3) By the Poincaré-Birkhoff-Witt theorem, R' is isomorphic to a polynomial ring. In particular R' is Noetherian. By the theorem of Eakin-Nagata and (1), we have R is Noetherian. This implies (3).

(4) Since $U(\lambda)$ is a finitely generated $U(\mathfrak{n})$-module and $U(\mathfrak{n})$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module, $U(\lambda)$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module. Hence $U(\lambda)_0$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module by (3). $\square$

We enumerate $W=\{w_1,w_2,\dots,w_r\}$ such that $\operatorname{Re} w_1\lambda\geq \operatorname{Re} w_2\lambda\geq\cdots\geq\operatorname{Re} w_r\lambda$.

Theorem 3.9. *There exist matrices $A\in M(1,r,\widehat{\mathcal{E}}(\mathfrak{n})_{2\mathcal{P}})$ and $B\in M(r,1,\widehat{\mathcal{E}}(\mathfrak{a}\oplus\mathfrak{n},\mathfrak{n})_{2\mathcal{P}})$ such that $v_\lambda = Bu_\lambda\in(\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})}U(\lambda))^r$ satisfies the following conditions:*

- *There exists a linear map $Q\colon\mathfrak{a}\to M(r,U(\mathfrak{n})_{2\mathcal{P}})$ such that*

$$\begin{cases}
Hv_\lambda = Q(H)v_\lambda \text{ for all } H\in\mathfrak{a},\\
Q(H)_{ii} = (\rho+w_i\lambda)(H) \text{ for all } H\in\mathfrak{a},\\
\text{if } w_i\lambda-w_j\lambda\notin 2\mathcal{P}^+ \text{ then } Q(H)_{ij}=0 \text{ for all } H\in\mathfrak{a},\\
\text{if } w_i\lambda-w_j\lambda\in 2\mathcal{P}^+ \text{ then } [H',Q(H)_{ij}] = (w_i\lambda-w_j\lambda)(H')Q(H)_{ij} \text{ for all } H,H'\in\mathfrak{a}.
\end{cases}$$

- *We have* $u_\lambda = Av_\lambda$.
- *Let* $(v_1, v_2, \dots, v_r) = v_\lambda$. *Then* $\{v_i \pmod{\mathfrak{n}U(\lambda)}\}$ *is a basis of* $U(\lambda)/\mathfrak{n}U(\lambda)$.

Proof. Let $u_1, u_2, \dots, u_N$ be generators of $U(\lambda)_0$ as a $U(\mathfrak{n})_{2\mathcal{P}}$-module. These are also generators of $U(\lambda)$ as a $U(\mathfrak{n})$-module by Lemma 3.7. We choose matrices $C = {}^t(C_1, C_2, \dots, C_N) \in M(N, 1, U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}})$ and $D = (D_1, D_2, \dots, D_N) \in M(1, N, U(\mathfrak{n})_{2\mathcal{P}})$ such that ${}^t(u_1, u_2, \dots, u_N) = Cu_\lambda$ and $u_\lambda = D\,{}^t(u_1, u_2, \dots, u_N)$.

Notice that $U(\mathfrak{n})_{2\mathcal{P}} + \mathfrak{n}U(\mathfrak{n}) = U(\mathfrak{n})$. By Lemma 3.7,

$$\begin{aligned}
U(\lambda)/\mathfrak{n}U(\lambda) &= (U(\mathfrak{n})/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})} U(\lambda) \\
&= (U(\mathfrak{n})/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})} U(\mathfrak{n}) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} U(\lambda)_0 \\
&= (U(\mathfrak{n})/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} U(\lambda)_0 \\
&= ((U(\mathfrak{n})_{2\mathcal{P}} + \mathfrak{n}U(\mathfrak{n}))/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} U(\lambda)_0 \\
&= (U(\mathfrak{n})_{2\mathcal{P}}/(\mathfrak{n}U(\mathfrak{n}) \cap U(\mathfrak{n})_{2\mathcal{P}})) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} U(\lambda)_0 \\
&= (U(\mathfrak{n})_{2\mathcal{P}}/(\mathfrak{n}U(\mathfrak{n}))_{2\mathcal{P}}) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} U(\lambda)_0 \\
&= U(\lambda)_0/(\mathfrak{n}U(\mathfrak{n}))_{2\mathcal{P}} U(\lambda)_0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
U(\lambda)/\mathfrak{n}U(\lambda) &= U(\mathfrak{g})/(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \\
&= (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{a}) + U(\mathfrak{g})\mathfrak{k})/(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \\
&= U(\mathfrak{a})/((\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \cap U(\mathfrak{a})).
\end{aligned}$$

By the definition of χ_λ, we have

$$(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \cap U(\mathfrak{a}) = \sum_{p \in U(\mathfrak{a})^W} U(\mathfrak{a})(\chi_2^{-1}(p) - \lambda(p))$$

where $U(\mathfrak{a})^W$ is a $\mathbb{C}$-algebra of W-invariant elements of $U(\mathfrak{a})$. By the result of Oshima [Osh88, Proposition 2.8], the set of eigenvalues of $H \in \mathfrak{a}$ on $U(\mathfrak{a})/(\sum_{p \in U(\mathfrak{a})^W} U(\mathfrak{a})(\chi_2^{-1}(p) - \lambda(p)))$ is $\{(\rho + w\lambda)(H) \mid w \in W\}$ with multiplicities.

We take matrices $A' \in M(N, r, \widehat{\mathcal{E}}(\mathfrak{n})_{2\mathcal{P}})$ and $B' \in M(r, N, \widehat{\mathcal{E}}(\mathfrak{n})_{2\mathcal{P}})$ such that the conditions of Theorem 3.1 hold. Put $A = DA'$, $B = B'C$ then A, B satisfy the conditions of the theorem. $\square$

§4. Structure of Jacquet modules (regular case)

In this section we assume that λ is regular, i.e., $w\lambda \neq \lambda$ for all $w \in W \setminus \{e\}$. Let $r = \#W$ and $v_\lambda = (v_1, v_2, \dots, v_r) \in (\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} U(\lambda))^r$ as in Theorem 3.9. Set $\mathcal{W}(i) = \{j \mid w_i\lambda - w_j\lambda \in 2\mathcal{P}^+\}$ for each $i = 1, 2, \dots, r$.

Theorem 4.1. *We have* $Xv_i \in \sum_{j \in \mathcal{W}(i)} U(\mathfrak{g})v_j$ *for all* $X \in \theta(\mathfrak{n}) \oplus \mathfrak{m}$.

Let $A = {}^t(A^{(1)}, A^{(2)}, \dots, A^{(r)})$ be as in Theorem 3.9 and $\overline{A} = {}^t(\overline{A^{(1)}}, \overline{A^{(2)}}, \dots, \overline{A^{(r)}})$ an element of $M(r, 1, \mathbb{C})$ such that $A^{(i)} - \overline{A^{(i)}} \in \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n})$.

Lemma 4.2. *We have* $\overline{A^{(i)}} \neq 0$ *for each* $i = 1, 2, \dots, r$.

Proof. Put $\overline{U(\lambda)} = U(\lambda)/\mathfrak{n}U(\lambda)$, $\overline{u_\lambda} = u_\lambda \pmod{\mathfrak{n}U(\lambda)}$ and $\overline{v_i} = v_i \pmod{\mathfrak{n}U(\lambda)}$. Let $\overline{B} = (\overline{B^{(1)}}, \overline{B^{(2)}}, \dots, \overline{B^{(r)}})$ be a matrix in $M(1, r, U(\mathfrak{a}))$ such that $\overline{v_j} = \overline{B^{(j)}}\,\overline{u_\lambda}$. Then we have $\overline{v_j} = \sum_i \overline{A^{(i)}}\,\overline{B^{(j)}}\,\overline{v_i}$. By the regularity of λ, we have $H\overline{v_j} = (w_j\lambda)(H)\overline{v_j}$ and $H\overline{B^{(j)}}\,\overline{v_i} = (w_i\lambda)(H)\overline{B^{(j)}}\,\overline{v_i}$ for all $H \in \mathfrak{a}$. This implies $\overline{A^{(j)}} \neq 0$ since λ is regular. $\square$

Proof of Theorem 4.1. Put $f(\mathbf{n}) = \sum_i \mathbf{n}_i\beta_i$ for $\mathbf{n} = (\mathbf{n}_i) \in \mathbb{Z}^m$. Set $\widetilde{\Lambda} = \{\mathbf{n} \in \mathbb{Z}^m_{\geq 0} \mid f(\mathbf{n}) \in 2\mathcal{P}\}$. We write $A^{(j)} = \sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} E^{\mathbf{n}}$. Let $\alpha \in \Sigma^+$ and $E_\alpha \in \mathfrak{g}_\alpha$. Since $\mathfrak{k}u_\lambda = 0$, we have $(\theta(E_\alpha) + E_\alpha)u_\lambda = 0$. Hence $(\theta(E_\alpha) + E_\alpha)\sum_j \sum_{\mathbf{n}} A^{(j)}_{\mathbf{n}} E^{\mathbf{n}} v_j = 0$.

By applying Corollary 2.9 we have

$$\sum_{j=1}^{r} \left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} (\theta(E_\alpha) + E_\alpha) E^{\mathbf{n}} \right)^{(w_i\lambda - w_j\lambda - \alpha)} v_j = 0$$

for $i = 1, 2, \dots, r$. On one hand if $w_i\lambda - w_j\lambda \notin 2\mathcal{P}_+$ then

$$\left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} (\theta(E_\alpha) + E_\alpha) E^{\mathbf{n}} \right)^{(w_i\lambda - w_j\lambda - \alpha)} = 0.$$

On the other hand

$$\left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(i)}_{\mathbf{n}} (\theta(E_\alpha) + E_\alpha) E^{\mathbf{n}} \right)^{(-\alpha)} = A^{(i)}_{\mathbf{0}}\theta(E_\alpha).$$

Hence we have

$$A^{(i)}_{\mathbf{0}}\theta(E_\alpha)v_i \in \sum_{j\in\mathcal{W}(i)} U(\mathfrak{g})v_j.$$

Since $A^{(i)}_{\mathbf{0}} = \overline{A^{(i)}} \neq 0$, we have

$$\theta(E_\alpha)v_i \in \sum_{j\in\mathcal{W}(i)} U(\mathfrak{g})v_j.$$

Next let X be an element of $\mathfrak{m}$. By Corollary 2.9, we have

$$\sum_{j=1}^{r} \left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} X E^{\mathbf{n}} \right)^{(w_i\lambda - w_j\lambda)} v_j = 0.$$

We can prove $Xv_i \in \sum_{j\in\mathcal{W}(i)} U(\mathfrak{g})v_j$ by the same argument. $\square$

Put $V(\lambda) = \sum_i U(\mathfrak{g})v_i \subset \widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} U(\lambda)$.

Corollary 4.3.

$$V(\lambda) = J(U(\lambda)).$$

Proof. By Theorem 4.1, $V(\lambda)$ is finitely generated as a $U(\mathfrak{n})$-module. By applying Proposition 2.4, we see that the map $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) \to \prod_{\mu\in\mathfrak{a}^*} V(\lambda)_\mu$ is bijective. Hence $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) \to \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U(\lambda)$ is injective by Proposition 2.8. This map is also surjective since $v_1, v_2, \ldots, v_r$ are generators of $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U(\lambda)$.

We have $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) = \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U(\lambda)$. Since $U(\lambda)$ and $V(\lambda)$ are finitely generated as $U(\mathfrak{n})$-modules, we have

$$\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} U(\lambda) = \widehat{J}(U(\lambda)),$$
$$\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) = \widehat{J}(V(\lambda)),$$

by Proposition 2.2. Hence we have $J(U(\lambda)) = J(V(\lambda)) = V(\lambda)$ by Corollary 2.6. □

Recall the definition of a generalized Verma module. Set $\overline{\mathfrak{p}} = \theta(\mathfrak{p})$, $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$ and $\rho = \sum_{\alpha\in\Sigma^+}(\dim \mathfrak{g}_\alpha/2)\alpha$.

Definition 4.4 (Generalized Verma module). *Let $\mu \in \mathfrak{a}^*$. Define the one-dimensional representation $\mathbb{C}_{\rho+\mu}$ of $\overline{\mathfrak{p}}$ by $(X+Y+Z)v = (\rho+\mu)(Y)v$ for $X \in \mathfrak{m}$, $Y \in \mathfrak{a}$, $Z \in \overline{\mathfrak{n}}$, $v \in \mathbb{C}_{\rho+\mu}$. We define a $U(\mathfrak{g})$-module $M(\mu)$ by*

$$M(\mu) = U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{p}})} \mathbb{C}_{\rho+\mu}.$$

This is called a generalized Verma module.

Set $V_i = \sum_{j\geq i} U(\mathfrak{g})v_j$. By the universality of tensor products, any $U(\overline{\mathfrak{p}})$-module homomorphism $\mathbb{C}_{\rho+\mu} \to U$ is uniquely extended to the $U(\mathfrak{g})$-module homomorphism $M(\mu) \to U$ for a $U(\mathfrak{g})$-module U. In particular we have the surjective $U(\mathfrak{g})$-module homomorphism $M(w_i\lambda) \to V_i/V_{i+1}$. We shall show that V_i/V_{i+1} is isomorphic to a generalized Verma module using the character theory.

Let G be a connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}_0$, K its maximal compact subgroup with its Lie algebra $\mathfrak{k}_0$, P the parabolic subgroup whose Lie algebra is $\mathfrak{p}_0$ and $P = MAN$ the Langlands decomposition of P where Lie algebra of M (resp. A, N) is $\mathfrak{m}_0$ (resp. $\mathfrak{a}_0$, $\mathfrak{n}_0$).

Since $U(w\lambda) = U(\lambda)$ for $w \in W$, we may assume that $\operatorname{Re}\lambda$ is dominant, i.e., $\operatorname{Re}\lambda(H_i) \leq 0$ for each $i = 1, 2, \ldots, l$. By the result of Kostant [Kos75, Theorem 2.10.3], $U(\lambda)$ is isomorphic to the space of K-finite vectors of the non-unitary principal series representation $\mathrm{Ind}_P^G(1 \otimes \lambda)_K$. The character of this representation is calculated by Harish-Chandra (See Knapp [Kna01, Proposition 10.18]). Before we state it, we prepare some notations. Let $H = TA$ be the maximally split Cartan subgroup, $\mathfrak{h}_0$ its Lie algebra, $T = H \cap M$, Δ the root system of H, Δ^+ the positive system compatible with Σ^+, Δ_I the set of imaginary roots, $\Delta_I^+ = \Delta^+ \cap \Delta_I$ and ξ_α the one-dimensional representation of H whose derivation is α for $\alpha \in \mathfrak{h}^*$. Under these notations, the distribution character $\Theta_G(U(\lambda))$ of $U(\lambda)$ is as follows;

$$\Theta_G(U(\lambda))(ta) = \frac{\sum_{w\in W} \xi_{\rho+w\lambda}(a)}{\prod_{\alpha\in\Delta^+\setminus\Delta_I^+} |1 - \xi_\alpha(ta)|} \quad (t \in T,\ a \in A).$$

We will use the Osborne conjecture, which was proved by Hecht and Schmid [HS83a, Theorem 3.6]. To state it, we must define a character of $J(U)$ for a Harish-Chandra module U. Recall that $J(U)$ is an object of the category $\mathcal{O}'_P$, i.e.,

(1) the actions of $M \cap K$ and $\mathfrak{g}$ are compatible,

(2) $J(U)$ splits under $\mathfrak{a}$ into a direct sum of generalized weight spaces, each of them being a Harish-Chandra modules for MA,

(3) $J(U)$ is $U(\overline{\mathfrak{n}})$- and $Z(\mathfrak{g})$-finite

(See Hecht and Schmid [HS83b, (34)Lemma]). For an object V of $\mathcal{O}'_P$, we define the character $\Theta_P(V)$ of V by

$$\Theta_P(V) = \sum_{\mu\in\mathfrak{a}^*} \Theta_{MA}(V_\mu),$$

where V_μ is a generalized μ-weight space of V. Let G' be the set of regular elements of G. Set

$$A^- = \{a \in A \mid \alpha(\log a) < 0 \text{ for all } \alpha \in \Sigma^+\},$$

$$(MA)^- = \text{interior of } \left\{ g \in MA \;\middle|\; \prod_{\alpha\in\Delta^+\setminus\Delta_I^+} (1 - \xi_\alpha(ga)) \geq 0 \text{ for all } a \in A^- \right\} \text{ in } MA.$$

Then the Osborn conjecture says that $\Theta_G(U)$ and $\Theta_P(J(U))$ coincide on $(MA)^- \cap G'$ (See Hecht and Schmid [HS83b, (42)Lemma]). It is easy to calculate the character of a generalized Verma module. We have

$$\Theta_P(M(\mu))(ta) = \frac{\xi_{\rho+\mu}(a)}{\prod_{\alpha\in\Delta^+\setminus\Delta_I^+}(1-\xi_\alpha(ta))} \quad (t \in T,\ a \in A).$$

Consequently we have

$$\Theta_P(J(U(\lambda))) = \sum_{w\in W} \Theta_P(M(w\lambda)).$$

This implies the following theorem when λ is regular.

Theorem 4.5. *There exists a filtration $0 = V_{r+1} \subset V_r \subset \cdots \subset V_1 = J(U(\lambda))$ of $J(U(\lambda))$ such that V_i/V_{i+1} is isomorphic to $M(w_i\lambda)$ for an arbitrary $\lambda \in \mathfrak{a}^*$. Moreover if $w\lambda - \lambda \notin 2\mathcal{P}$ for all $w \in W \setminus \{e\}$ then $J(U(\lambda)) \simeq \bigoplus_{w\in W} M(w\lambda)$.*

§5. Structure of Jacquet modules (singular case)

In this section, we shall prove Theorem 4.5 in the singular case using the translation principle. We retain notations in Section 4. Let λ' be an element of $\mathfrak{a}^*$ such that following conditions hold:

- The weight λ' is regular.
- The weight $(\lambda - \lambda')/2$ is integral.
- The real part of λ' belongs to the same Weyl chamber which real part of λ belongs to.

First we define the translation functor $T_{\lambda'}^{\lambda}$. Let U be a $U(\mathfrak{g})$-module which has an infinitesimal character λ'. (We regard $\mathfrak{a}^* \subset \mathfrak{h}^*$.) We define $T_{\lambda'}^{\lambda}(U)$ by $T_{\lambda'}^{\lambda}(U) = P_\lambda(U \otimes E_{\lambda-\lambda'})$ where:

- $E_{\lambda-\lambda'}$ is the finite-dimensional irreducible representation of $\mathfrak{g}$ with an extreme weight $\lambda-\lambda'$.
- $P_\lambda(V)=\{v\in V \mid \text{for some } n>0 \text{ and all } z\in Z(\mathfrak{g}),\ (z-\lambda(\widetilde{\chi}(z)))^n v=0\}$ where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ and $\widetilde{\chi}\colon Z(\mathfrak{g})\to U(\mathfrak{h})$ is the Harish-Chandra homomorphism.

Notice that P_λ and $T_{\lambda'}^{\lambda}$ are exact functors. Theorem 4.5 in the singular case follows from following two equations.

(1) $T_{\lambda'}^{\lambda}(U(\lambda'))=U(\lambda)$.

(2) $T_{\lambda'}^{\lambda}(M(w\lambda'))=M(w\lambda)$.

The following lemma is important to prove these equations.

Lemma 5.1. *Let ν be a weight of $E_{\lambda-\lambda'}$ and $w\in W$. Assume $\nu=w\lambda-\lambda'$. Then $\nu=\lambda-\lambda'$.*

Proof. See Vogan [Vog81, Lemma 7.2.18]. □

Proof of $T_{\lambda'}^{\lambda}(U(\lambda'))=U(\lambda)$. We may assume that λ' is dominant. Notice that we have $U(\lambda')\simeq \mathrm{Ind}_P^G(1\otimes\lambda')_K$. Let $0=E_0\subset E_1\subset E_2\subset\cdots\subset E_n=E_{\lambda-\lambda'}$ be a P-stable filtration with the trivial induced action of N on E_i/E_{i-1}. We may assume that E_i/E_{i-1} is irreducible. Let ν_i be the highest weight of E_i/E_{i-1}. Notice that $\mathrm{Ind}_P^G(1\otimes\lambda')\otimes E_{\lambda-\lambda'}=\mathrm{Ind}_P^G((1\otimes\lambda')\otimes E_{\lambda-\lambda'})$. Then $\mathrm{Ind}_P^G(1\otimes\lambda')\otimes E_{\lambda-\lambda'}$ has a filtration $\{M_i\}$ such that $M_i/M_{i-1}\simeq \mathrm{Ind}_P^G((1\otimes\lambda')\otimes(E_i/E_{i-1}))$. Since $\mathrm{Ind}_P^G((1\otimes\lambda')\otimes(E_i/E_{i-1}))$ has an infinitesimal character $\lambda+\nu_i$, $P_\lambda(M_i/M_{i-1})=0$ if $\nu_i\neq w\lambda-\lambda'$ for all $w\in W$. By Lemma 5.1 we have $T_{\lambda'}^{\lambda}(\mathrm{Ind}_P^G(1\otimes\lambda')=\mathrm{Ind}_P^G((1\otimes\lambda')\otimes(E_i/E_{i-1}))$ where $\nu_i=\lambda-\lambda'$. By the conditions of λ', the action of M on $(\lambda-\lambda')$-weight space of $E_{\lambda-\lambda'}$ is trivial. Consequently $T_{\lambda'}^{\lambda}(\mathrm{Ind}_P^G(1\otimes\lambda'))=\mathrm{Ind}_P^G((1\otimes\lambda')\otimes(\lambda-\lambda'))=\mathrm{Ind}_P^G(1\otimes\lambda)$. □

Proof of $T_{\lambda'}^{\lambda}(M(w\lambda'))=M(w\lambda)$. We may assume $w=e\in W$. Since $M(\lambda')\otimes E_{\lambda-\lambda'}=U(\mathfrak{g})\otimes(\mathbb{C}_{\lambda'}\otimes E_{\lambda-\lambda'})$, the equation follows by the same argument of the proof of $T_{\lambda'}^{\lambda}(U(\lambda'))=U(\lambda)$. □

References

[Cas80] W. Casselman, *Jacquet modules for real reductive groups*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978) (Helsinki), Acad. Sci. Fennica, 1980, pp. 557–563.

[Col91] David H. Collingwood, *Jacquet modules for semisimple Lie groups having Verma module filtrations*, J. Algebra **136** (1991), no. 2, 353–375.

[GW80] Roe Goodman and Nolan R. Wallach, *Whittaker vectors and conical vectors*, J. Funct. Anal. **39** (1980), no. 2, 199–279.

[HS83a] Henryk Hecht and Wilfried Schmid, *Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules*, Acta Math. **151** (1983), no. 1-2, 49–151.

[HS83b] Henryk Hecht and Wilfried Schmid, *On the asymptotics of Harish-Chandra modules*, J. Reine Angew. Math. **343** (1983), 169–183.

[Kna01] Anthony W. Knapp, *Representation theory of semisimple groups*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001, An overview based on examples, Reprint of the 1986 original.

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.

[KO77] Masaki Kashiwara and Toshio Oshima, *Systems of differential equations with regular singularities and their boundary value problems*, Ann. of Math. (2) **106** (1977), no. 1, 145–200.

[Kos75] Bertram Kostant, *On the existence and irreducibility of certain series of representations*, Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), Halsted, New York, 1975, pp. 231–329.

[Lep75] J. Lepowsky, *Existence of conical vectors in induced modules*, Ann. of Math. (2) **102** (1975), no. 1, 17–40.

[Osh84] Toshio Oshima, *Boundary value problems for systems of linear partial differential equations with regular singularities*, Group representations and systems of differential equations (Tokyo, 1982), Adv. Stud. Pure Math., vol. 4, North-Holland, Amsterdam, 1984, pp. 391–432.

[Osh88] Toshio Oshima, *A realization of semisimple symmetric spaces and construction of boundary value maps*, Representations of Lie groups, Kyoto, Hiroshima, 1986, Adv. Stud. Pure Math., vol. 14, Academic Press, Boston, MA, 1988, pp. 603–650.

[SW82] J. T. Stafford and N. R. Wallach, *The restriction of admissible modules to parabolic subalgebras*, Trans. Amer. Math. Soc. **272** (1982), no. 1, 333–350.

[Vog81] David A. Vogan, Jr., *Representations of real reductive Lie groups*, Progress in Mathematics, vol. 15, Birkhäuser Boston, Mass., 1981.